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Layouts with Wires of Balanced Length

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For any graph G with fixed boundary there exists a layout in the plane, which minimizes the maximum Euclidean distance of any node to its neighbors. This layout balances the length of the graph edges and is therefore called a (length-) balanced layout of G. Furthermore the existence of a unique optimal balanced layout L with the following properties is proved: (i) L is the minimal element of an order defined on the set of layouts of a graph with fixed boundary. (ii) L may be constructed as the limit of the l_p-optimal layouts L_p of G. (iii) If G is a planar graph with fixed boundary, then the optimal balanced layout L of G is quasi-planar.

INTRODUCTION

In this paper we consider graphs with fixed boundaries and present some results on (optimal) balanced layouts of those graphs in the Euclidean plane.

In the area of design automation a graph is a commonly used model to represent the abstract structure of a switching circuit (nodes of the graph = modules of the circuit, edges of the graph = signal nets interconnecting the modules). In any realization of such a circuit on a (VLSI-) chip, there are some modules, the input- and output-pins, which have to be located on the boundary of the chip area. In our model we therefore assume, that a cycle of the graph is given and embedded in the plane which corresponds to the boundary of the chip and whose nodes are identified with the i/o-pins of the circuit. We call this structure a "graph with fixed boundary." So a graph with fixed boundary models the abstract structure of a circuit together with topological constraints given by the boundary of the chip and fixed i/o-ports.

For the actual construction of a circuit on a chip it is necessary to place the (remaining) modules or function blocks on the chip area in such a way, that the wires interconnecting corresponding ports can be routed according to given electrical constraints and optimization criteria. These problems are well known as placement and routing problems (Breuer, 1972). In the graph model the solution of a placement or routing problem corresponds

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to the construction of an embedding with respect to a given cost function. In this sense a layout of a graph with fixed boundary in the Euclidean plane within the area prescribed by the fixed boundary may be interpreted as a first placement and global routing of the corresponding circuit. This is done (and the results of this paper are used) in a CAD-system for the design of integrated circuits at the Universität des Saarlandes. (For detailed information see Hotz *et al.*, 1986.)

There are a lot of different cost functions which have been considered; for this paper we concentrate on the "minimization of the length of the longest edge" in the layout, and at the same time we try to "balance the length of the remaining edges" (i.e., not only the length of the longest edge but also the length of the remaining edges will be as short as possible). This is motivated by the following: The interconnections between the modules should be as short as possible, since long wires result in large capacitances and large capacitances require large transistors to drive them, in order to avoid long signal delays.

Similar optimization goals have been considered by many authors (e.g., Bhatt and Leiserson, 1982; Patterson *et al.*, 1981; Leighton, 1981; Blum, 1984). All these papers have the following in common: they consider embeddings of graphs (resp. restricted classes of graphs, e.g., binary trees) and present (upper and lower) bounds on the area and the length of the longest edge with respect to the "rectangular grid model." (I.e., in general only the length l of the longest edge is minimized, the length of the remaining edges is not considered as long as it is less or equal than l; sometimes area (resp. wire-length) is minimized under the constraint that all wires must have the same length.)

We consider (planar) embeddings in a continuous part of the plane (not within a rectangular grid) and the resulting layouts may have wires running in any direction of the plane. This is not possible in current technologies, but there seems to be no physical necessity to demand rectangular routings. Efforts to generalize the concept of strictly horizontal or vertical wires can be observed (Bryant, 1983).

There is no doubt, that a graph with fixed boundary is a very idealized model of a physical circuit: nodes and edges have no realistic dimensions and may be located arbitrarily close together or even collapse on each other. We show in this paper, that in the "neighborhood" of any optimal layout there exists a layout without collapsed nodes or edges. In order to construct a realistic physical circuit, such a layout of the idealized graph has to be transformed into a layout which realizes the edges and nodes by geometric configurations of given dimensions and thus separates edges and nodes according to given design rules. This can be done with the help of an algorithm called "logarithmic pumping," which assigns the necessary configurations to nodes and edges on one side and minimizes the layout area in an iterative process on the other side. For detailed information see (Becker *et al.*, 1984; Schworm, 1985).

In this paper we restrict ourselves to the following questions: Consider a graph G with fixed boundary.

—Does there exist a layout L of G such that the maximum distance of any node (not only the node with the longest edge) to its neighbors is minimal?

—How can this layout be constructed?

A "local version" of this problem is well known in the area of geometrical algorithms:

> Given a finite set S of points in the plane. Find the point s in the plane, which minimizes the maximum distance to the points of S.

For the solution of this problem efficient algorithms are known (Shamos and Hoey, 1975; Osthof, 1983). They use Voronoi diagrams to find the point s in time $O(n \log n)$ and space $O(n)$ ($n = \mathrm{card}(S)$).

A first hint, that the "global version" of this problem has a solution is given in Becker and Hotz (1983). There optimal layouts of graphs with fixed boundary, which minimize the sum of the pth power of the edge length, are investigated. From elementary analysis one concludes that the sequence $(L_p)_{p \in N}$ of the l_p-optimal layouts of a graph with fixed boundary approximates a layout L minimizing the maximum length of an edge in the layout.

In the present paper we show the existence of a layout which minimizes the maximum distance of any node to its neighbors (= "balanced" layout) in the following way:

We introduce an order on the set of layouts of a graph with fixed boundary and show the existence and uniqueness of the minimal element. This minimal element L turns out to be a balanced layout, which is the limit of the sequence $(L_p)_{p \in N}$. L is called an optimal balanced layout because of the minimality. Finally, we conclude from Becker and Hotz (1983): if G is a planar graph with fixed boundary, then the optimal balanced layout is quasi-planar.

Not only the existence and properties of (optimal) balanced layouts are interesting but also their construction. Since the optimal balanced layout is the limit of the l_p-optimal layouts, a layout L_p for p sufficiently large is a good approximation for L. In the field of numerical analysis the following is commonly accepted: L_p for $8 \leqslant p \leqslant 12$ gives a sufficiently good approximation for the optimal balanced layout. (This is confirmed by our experimental results, see, e.g. Fig. 6.) In this sense approximation algorithms for l_p-optimal layouts are important.

There exist efficient polynomial-time algorithms in the model of "graph with fixed boundary" (in contrast to the grid model, where corresponding problems such as the "Quadratic Assignment Problem" are NP-hard (see Sahni and Gonzalez, 1976): In the case $p=2$ the construction of the l_2-optimal layout is equivalent to the solution of two sparse linear equation systems. In the case $p>2$ methods of "steepest descent" have been applied successfully. For more details see Groh (1983). In the case $p=2$, "multigrid methods" even allow solution of the optimization problem in linear time (Stüben, 1983). If this can be extended to $p>2$ is not known at this time. The approximation algorithms were tested for a lot of (complex) circuits (e.g., carry lookahead adder, conditional sum adder, multiplier, memories, combinational logic,...). In Fig. 6b we give the optimal balanced layout of a fast multiplier presented in (Luk and Vuillemin, 1983) (the fixed boundary is omitted for simplicity). The multiplier can be described with the help of four recursive equations in a very comfortable way (for details see Becker *et al.*, 1984). Here we start directly with the graph with fixed boundary constructed from the equations (see Fig. 6a).

Basic Definitions and Remarks

Let $G=(V, E)$ be an undirected connected simple graph. A layout L of G in the plane is given by a mapping $L\colon V \to R^2$; i.e., the edges $e=(v_1, v_2)$ of G are mapped on the straight line segments $\overline{L(v_1)\,L(v_2)} = :L(e)$.

$G_{R(c)} := (G, c, R)$ is called a **graph with fixed boundary**, iff the following holds:

—$G=(V, E)$ is a (undirected connected simple) graph with $V = V_{\text{in}} \cup V_c$.

—$c=(V_c, E_c)$ is a nontrivial cycle of G without double points.

—$R\colon V_c \to R^2$ is a layout of c, which maps c onto a convex polygon $R(c)$.

A layout L of G (not.: $L(G)$) is called **R(c)-respecting**, iff $G_{R(c)}=(G, c, R)$ is a graph with fixed boundary and $L|c=R$, i.e., the cycle c is mapped on $R(c)$ in the layout L. We give an example of an $R(c)$-respecting layout in Fig. 1.

Henceforth we only consider layouts of G, which are $R(c)$-respecting; $L(G, c, R) := \{L | L \text{ is a } R(c)\text{-respecting layout of } G\}$ is called the set of **R(c)-respecting layouts of *G***.

We are interested in particular elements of $L(G, c, R)$: the balanced layouts. For $v \in V$ the edge set of v is defined as $E(v) := \{e | e \in E, \exists w \in V\colon e=(v, w)\}$. Let L be a layout of G. Then the edge set of v is **balan-**

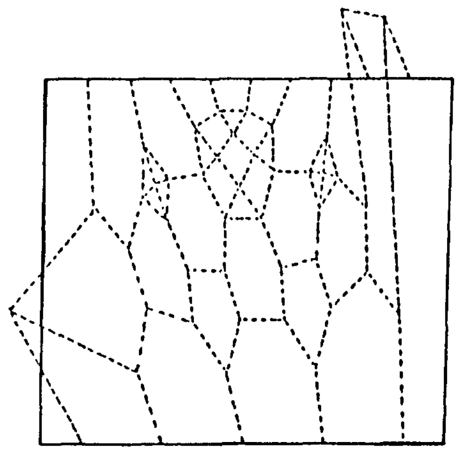

FIG. 1. $R(c)$-respecting layout ("——" denotes the fixed boundary, "- - -" the remaining edges).

ced in L (= the **minmax condition** holds for $L(v)$), iff $\max\{|L(e)| \mid e \in E(v)\}$ is minimal for all possible locations $L(v)$.

L is called a **balanced layout**, iff $\forall v \in V_{\text{in}}$: the edge set of v is balanced in L. Until now we do not have the existence of a balanced layout for any graph G, but we get the following:

—If L is a balanced layout, then L is **R(c)-bounded**, i.e., $\forall v \in V_{\text{in}}$ we have: $L(v)$ lies inside or on $R(c)$. (Fig. 1 gives an example of a layout L, which is not balanced, since L is not $R(v)$-bounded.)

—There exist graphs with balanced layouts: e.g., if $|V_{\text{in}}| = 1$, then the balanced layout is unique and can be constructed in time $O(n \log n)$ (Shamos and Hoey, 1975). If $|V_{\text{in}}| > 1$, there may exist continuously many balanced layouts (see Fig. 2). In this example the symmetric layout is the optimal balanced layout, since max $\{|L(e)| \mid e \in E \backslash E_c\}$ is minimal.

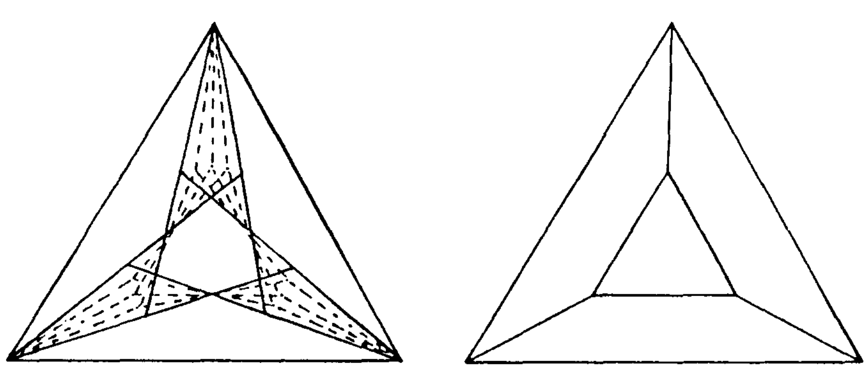

FIGURE 2

In the following we develop a method to find the optimal balanced layout in general. At first we define:

Let L be a layout of G, $E_{\text{in}} := E \backslash E_c$. We partition E_{in} into classes of edges of equal length (in L):

$\text{Sk}(1) := \{e \mid L(e)$ has maximum length in $L(E_{\text{in}})\}$ is called **skeleton 1**.

Inductively we define for $\mathbf{i} \in \mathbf{N}$ and $E_{\text{in}}(i) := E_{\text{in}} \backslash \bigcup_{j=1}^{i-1} \text{Sk}(j) \neq \varnothing$:

$\text{Sk}(i) := \{e \mid L(e)$ has maximum length in $L(E_{\text{in}}(i))\}$ is called **skeleton i**.

If we consider skeleton i of a layout L_1 and skeleton i of a layout L_2, we write $\text{Sk}(i)(L_1)$ (resp. $\text{Sk}(i)(L_2)$).

Now we use skeletons to define an order on $L(G, c, R)$, which will help us to find a balanced layout. This balanced layout will be a minimal element of the order and in this sense "optimally balanced."

A layout L of G is classified by the number

$$N(L) := (l_1, n_1, l_2, n_2, \ldots, l_k, n_k)$$

with

$$\begin{aligned} k &:= \text{number of skeletons in } L \\ l_i &:= |L(e)| \text{ with } e \text{ in } \text{Sk}(i) \\ n_i &:= \text{card}(\text{Sk}(i)). \end{aligned}$$

$N^j(L)$ is defined to be the first $2j$ components of $N(L)$ for $1 \leqslant j \leqslant k$.

Let L_1, L_2 be two layouts in $L(G, c, R)$. Then the following order "$<$" with respect to the lexicographical order in the components of N is well defined:

$$L_1 < L_2 :\Leftrightarrow N(L_1) < N(L_2).$$

From the definition of balanced and "$<$" we get at once

LEMMA 1. *If L is a minimal element of "$<$," then L is a balanced layout.*

Lemma 1 justifies the following notation: A minimal element of "$<$" is called an **optimal balanced layout**. An intuitive interpretation of the definition of optimal balanced layout is given by

> The edges of skeleton 1 are realized with minimum edge length, additionally the number of edges in $\text{Sk}(1)$ is minimal.
> Iteratively, skeleton i is embedded in $\bigcup_{j=1}^{i-1} L(\text{Sk}(j) \cup R(c))$ with minimal edge length and minimal edge number.

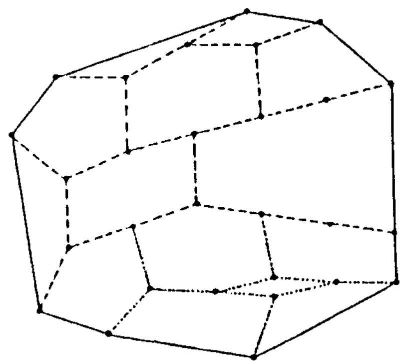

FIG. 3. Example of a (optimal balanced) layout with two skeletons, where "——" denotes $R(c)$, "- - -" edges of Sk(1), "···" edges of Sk(2).

(Figure 3 gives an example of an optimally balanced layout with two skeletons.)

THEOREMS AND PROOFS

For the proof of the existence and uniqueness of the optimal balanced layout the following method to construct a new layout starting from two different layouts L_1, L_2 is essential.

Let L_1, L_2 be two layouts of G. A layout L of G is called an (L_1, L_2)-**layout**, iff L is defined as follows: $\forall v \in V_{\text{in}}: L(v)$ bisects the straight line $\overline{L_1(v)\, L_2(v)}$. The (L_1, L_2)-layout is the "average"-layout of L_1 and L_2. We formalize this in the subsequent Facts 1 and 2.

FACT 1. *Consider an edge $e \in E$, layouts L_1, L_2 of G, and the (L_1, L_2)-layout L. Then we have:*

(i) $|L(e)| \leqslant (|L_1(e)| + |L_2(e)|)/2$

(ii) $|L(e)| = (|L_1(e)| + |L_2(e)|)/2 \Leftrightarrow L_1(e), L_2(e)$ *are collinear.*

We omit the easy proof of Fact 1.

Assume L_1, L_2 are two different layouts wth $N(L_1) \geqslant N(L_2)$, $(N(L_i) = (l_1(L_i), n_1(L_i), ..., l_{k_i}(L_i), n_{k_i}(L_i)))$, and define

$$m := \begin{cases} 0 & \text{iff} \quad N^1(L_1) \neq N^1(L_2) \\ \max_{j \in N} \{N^j(L_1) = N^j(L_2)\} & \text{otherwise.} \end{cases}$$

Now choose $t \in [0, m]$ maximal with

$$\forall i,\ 1 \leqslant i \leqslant t\colon \begin{cases} l_i(L_1) = l_i(L_2) = l_i(L) \\ n_i(L_1) = n_i(L_2) = n_i(L). \end{cases} \tag{$*$}$$

(Exactly the first $2m$ components of $N(L_1)$, $N(L_2)$ are equal; exactly the first $2t$ components of $N(L_1)$, $N(L_2)$, and $N(L)$ are equal.) In Fact 2 we conclude that L is equal to L_1 and L_2 in the first t skeletons up to translations in the plane. The "first essential difference" occurs in skeleton $t+1$. More precisely:

FACT 2. (i) $\forall i,\quad 1 \leqslant i \leqslant t$: $\quad \mathrm{Sk}(i)(L_1) = \mathrm{Sk}(i)(L_2) = \mathrm{Sk}(i)(L)$ *and* $\forall e \in \mathrm{Sk}(i)(L_1)$: $|L_1(e)| = |L_2(e)| = |L(e)|$ *and* $L_1(e)$, $L_2(e)$, $L(e)$ *are collinear.*

(ii) $t < k$ *and either*: $l_{t+1}(L_1) > l_{t+1}(L)$

$$\text{or} \begin{cases} l_{t+1}(L_1) = l_{t+1}(L) \\ n_{t+1}(L_1) > n_{t+1}(L) \textit{ and} \\ \mathrm{Sk}(t+1)(L_1) \supset \mathrm{Sk}(t+1)(L) \textit{ and} \\ \forall e \in \mathrm{Sk}(t+1)(L)\colon \\ |L_1(e)| = |L(e)| \textit{ and} \\ L_1(e), L(e) \textit{ are collinear.} \end{cases}$$

Proof. (i) follows directly from $(*)$ and Fact 1. (ii) is proved in three cases.

Case 1. $t+1 \leqslant m$. This means $l_{t+1}(L_1) = l_{t+1}(L_2)$, $n_{t+1}(L_1) = n_{t+1}(L_2)$. We conclude from Fact 1 and (i), that either

$$l_{t+1}(L_1) \neq l_{t+1}(L) \Rightarrow l_{t+1}(L_1) > l_{t+1}(L)$$

or

$$\begin{Bmatrix} l_{t+1}(L_1) = l_{t+1}(L) \\ n_{t+1}(L_1) \neq n_{t+1}(L) \end{Bmatrix} \Rightarrow \begin{cases} l_{t+1}(L_1) = l_{t+1}(L) \\ n_{t+1}(L_1) > n_{t+1}(L) \quad \text{and} \\ \mathrm{Sk}(t+1)(L_1) \supset \mathrm{Sk}(t+1)(L) \quad \text{and} \\ \forall e \in \mathrm{Sk}(t+1)(L)\colon \\ |L_1(e)| = |L(e)| \quad \text{and} \\ L_1(e), L(e) \quad \text{are} \quad \text{collinear.} \end{cases}$$

Case 2. $t = m < k$. We conclude from Fact 1 and (i), that either

$$l_{m+1}(L_1) > l_{m+1}(L_2) \Rightarrow l_{t+1}(L_1) > l_{t+1}(L)$$

or

$$\left.\begin{array}{l} l_{m+1}(L_1) = l_{m+1}(L_2) \\ n_{m+1}(L_1) > n_{m+1}(L_2) \end{array}\right\} \Rightarrow \begin{array}{c} l_{t+1}(L_1) > l_{t+1}(L) \\ \text{or} \\ \left\{\begin{array}{l} l_{t+1}(L_1) = l_{t+1}(L) \\ n_{t+1}(L_1) > n_{t+1}(L) \quad \text{and} \\ \mathrm{Sk}(t+1)(L_1) \supset \mathrm{Sk}(t+1)(L) \quad \text{and} \\ \forall e \in \mathrm{Sk}(t+1)(L): \\ |L_1(e)| = |L(e)| \quad \text{and} \\ L_1(e), L(e) \quad \text{are} \quad \text{collinear.} \end{array}\right. \end{array}$$

Case 3. $t = m = k$. Then we have from (i) $\forall e \in E$: $|L_1(e)| = |L(e)| = |L_2(e)|$ and $L_1(e)$, $L(e)$, $L_2(e)$ are collinear. This is only possible for $L_1 = L = L_2$. So we have a contradiction to the assumption, that L_1, L_2 are different layouts. ▌

An easy consequence of Fact 2 is

LEMMA 2. *Let L_1, L_2 be two different $R(c)$-bounded layouts of G. If L is the (L_1, L_2)-layout, then $N(L) < \max\{N(L_1), N(L_2)\}$.*

As a consequence of Lemma 2 we get at once: If there exists a minimal element of "$<$," then this element is uniquely determined.

Next we concentrate on the existence proof of an optimally balanced layout. For this we repeat some results of Becker and Hotz (1983): In this paper l_p-optimal layouts of G are considered. L is l_p-optimal, iff $|L|_p := \sum_{e \in E} |L(e)|^p$ is minimal in $L(G, c, R)$, i.e., the sum of the pth power of the edge length is minimized. It is shown, that there exists a unique l_p-optimal layout L_p for all $p \geqslant 2$ and that these layouts are $R(c)$-bounded. Thus with the compactness of the set of points inside $R(c)$ we get

FACT 3. *Let $(L_p)_{p \in \mathbb{N}}$ be the sequence of l_p-optimal layouts of G. Then there exists a subsequence $(L_{p_i})_{i \in \mathbb{N}}$, which converges pointwise to a layout L of G.*

We are now ready to give the key lemma of this paper.

LEMMA 3. *Let L be the limit of the subsequence $(L_{p_i})_{i \in \mathbb{N}}$ of the l_{p_i}-optimal layouts of G. Then L is a minimal element of "$<$."*

Proof. Let $(L_{p_i})_{i \in \mathbb{N}}$ *be a subsequence with* $\lim_{i \to \infty} L_{p_i} = L$. We assume, that L is not minimal, i.e., $\exists L_0$, layout of G, with $N(L_0) < N(L)$. Let $\bar{L}$ be the (L, L_0)-*layout* of G.

The idea of the proof is as follows: With the help of $\bar{L}$ and Fact 2 we find an $i \in \mathbf{N}$, which allows the construction of a layout $\bar{L}_{p_i}$ (very similar to $\bar{L}$) with $|\bar{L}_{p_i}|_{p_i} < |L_{p_i}|_{p_i}$. This is a contradiction to the optimality of L_{p_i}. Thus L is minimal.

Now we give the exact proof: According to Fact 2, $\bar{L}$ has the following properties: $\exists t \in \mathbf{N}_0$ with

(i) $\forall j$, $1 \leqslant j \leqslant t$: $\bar{L}(\mathrm{Sk}(j))$ is a translation in the plane of $L(\mathrm{Sk}(j))$

(ii) either $l_{t+1}(L) > l_{t+1}(\bar{L})$ or $\bar{L}(\mathrm{Sk}(t+1)(\bar{L}))$ is a translation in the plane of a part of $L(\mathrm{Sk}(t+1)(L))$.

This means that layout $\bar{L}$ is equal to layout L in the first t skeletons up to translations. The "first difference" occurs in skeleton $t+1$. Define

$$\varepsilon := \begin{cases} l_{t+1}(L) - l_{t+1}(\bar{L}) & \text{if } \; l_{t+1}(L) \neq l_{t+1}(\bar{L}) \\ l_{t+1}(L) - l_{t+2}(\bar{L}) & \text{if } \; l_{t+1}(L) = l_{t+1}(\bar{L}). \end{cases}$$

Then ε is the difference between the longest edges in both layouts, which do not have equal length. Now choose δ small enough (e.g., $10\delta < \varepsilon$). Because $\lim_{i \to \infty} L_{p_i} = L$ there exists

$$I_1 \in \mathbf{N}: \forall i \geqslant I_1 \text{ and } \forall v \in V_{\text{in}}: |L_{p_i}(v) - L(v)| \leqslant \delta/2.$$

Let $N = \mathrm{card}(E)$; then there exists

$$I_2 \in \mathbf{N}: \forall q \geqslant I_2: (l_{t+1}(L) - \delta)^q > N \cdot (\bar{l})^q \tag{$**$}$$

with

$$\bar{l} := \begin{cases} l_{t+1}(\bar{L}) + \delta & \text{if } \; l_{t+1}(L) \neq l_{t+1}(\bar{L}) \\ l_{t+2}(\bar{L}) + \delta & \text{if } \; l_{t+1}(L) = l_{t+1}(\bar{L}). \end{cases}$$

Let $i \geqslant \max\{I_1, I_2\}$. Define $\bar{L}_{p_i}$ in the following way (for an illustration see Fig. 4): Consider for $j = 1, ..., t$,

$$E_j := \mathrm{Sk}(j)(L) \quad (= \mathrm{Sk}(j)(\bar{L}))$$

$$V_j := \{v \in V \mid \exists e \in \mathrm{Sk}(j)(L): e = (v, w)\}$$

$$V_{t+1} := \begin{cases} \phi & \text{if } \; l_{t+1}(L) \neq l_{t+1}(\bar{L}) \\ \{v \in V \mid \exists e \in \mathrm{Sk}(t+1)(\bar{L}): e = (v, w)\} & \text{otherwise} \end{cases}$$

$$E_{t+1} := \begin{cases} \phi & \text{if } \; l_{t+1}(L) \neq l_{t+1}(\bar{L}) \\ \mathrm{Sk}(t+1)(\bar{L}) & \text{otherwise.} \end{cases}$$

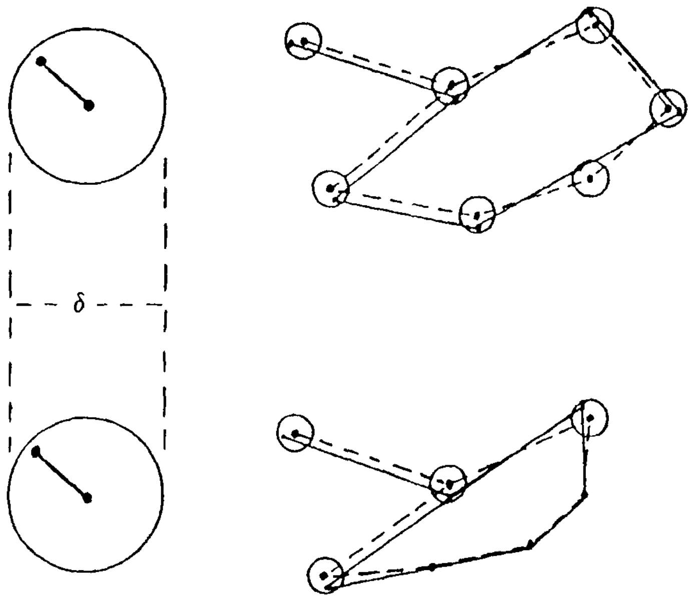

FIG. 4. $\bar{L}_{p_t}$ consists of three different types of edges: $\bar{L}_{p_t}$ is equal to L_{p_t} (up to translations) for all edges corresponding to the first $t+1$ (resp. t if $E_{t+1}=\emptyset$), skeletons of $\bar{L}$ (in the figure, edges between two δ-circles $=\tilde{E}$); those edges of $E\setminus\tilde{E}$, which have a common endpoint with edges of E, are lengthened at most by δ in comparison with the corresponding edges in $\bar{L}$ (in the figure, edges with one end in a δ-circle); the remaining edges $e\in E\setminus\tilde{E}$ of $\bar{L}_{p_t}$ are equal to the corresponding edges in $\bar{L}$ ($\bar{L}_{p_t}(e)=\bar{L}(e)$, in the figure, edges without ends in δ-circles). In the top part "——" denotes edges in L_{p_t} and "- - -" denotes edges in L. In the bottom part "——" denotes edges in $\bar{L}_{p_t}$ and "- - -" denotes edges in $\bar{L}$.

Define

$$\tilde{V} := \bigcup_{j=1}^{t+1} V_j$$

$$\tilde{E} := \bigcup_{j=1}^{t+1} E_j$$

$$\bar{L}_{p_t}(v) := \begin{cases} \bar{L}(v) & \forall v \in V\setminus\tilde{V} \\ \bar{L}(v) + (L_{p_t}(v) - L(v)) & \text{otherwise.} \end{cases}$$

If we now compare $|L_{p_t}|_{p_t}$ and $|\bar{L}_{p_t}|_{p_t}$, we get a contradiction to the optimality of L_{p_t}:

$$\begin{aligned} |L_{p_t}|_{p_t} &= \sum_{e\in\tilde{E}} |L_{p_t}(e)|^{p_t} + \sum_{e\in E\setminus\tilde{E}} |L_{p_t}(e)|^{p_t} \\ &\overset{\text{def of } \bar{L}_{p_t}}{=} \sum_{e\in\tilde{E}} |\bar{L}_{p_t}(e)|^{p_t} + \sum_{e\in E\setminus\tilde{E}} |L_{p_t}(e)|^{p_t} \end{aligned}$$

$$
\begin{aligned}
&\overset{\max_{e \in E \setminus \tilde{E}}\{|L_{p_i}(e)|\} \geqslant l_{t+1}(L) - \delta}{\geqslant} && \sum_{e \in \tilde{E}} |\bar{L}_{p_i}(e)|^{p_i} + (l_{t+1}(L) - \delta)^{p_i} \\
&\overset{(**)}{>} && \sum_{e \in \tilde{E}} |\bar{L}_{p_i}(e)|^{p_i} + N \cdot (\bar{l})^{p_i} \\
&\overset{\bar{l} = \max_{e \in E \setminus \tilde{E}}\{|\bar{L}(e)|\} + \delta}{\geqslant} && \sum_{e \in \tilde{E}} |\bar{L}_{p_i}(e)|^{p_i} + \sum_{e \in E \setminus \tilde{E}} (|\bar{L}(e)| + \delta)^{p_i} \\
&\geqslant && \sum_{e \in \tilde{E}} |\bar{L}_{p_i}(e)|^{p_i} + \sum_{e \in E \setminus \tilde{E}} |\bar{L}_{p_i}(e)|^{p_i} \\
&= && |\bar{L}_{p_i}|_{p_i} \qquad \text{contradiction!}
\end{aligned}
$$

We summarize our results in

THEOREM 1 (Existence and Uniqueness). (i) *The optimal balanced layout of G exists and is unique.*

(ii) $(L_p)_{p \in N}$ *converges to the optimal balanced layout* $L =: L_{\text{bal}}^{\text{opt}}$ *of* G.

Proof. (i) Follows directly from Lemmas 2 and 3.

(ii) With (i) and Lemma 3 we get that any subsequence of $(L_{p_i})_{i \in N}$ converges to L, i.e., $(L_p)_{p \in N}$ converges.

This gives the complete proof of Theorem 1. ∎

Let G be a $R(c)$-planar graph, i.e., there exists a layout L of G, which is $R(c)$-bounded and planar. From Becker and Hotz (1983) one knows that L_p is quasi-$R(c)$-planar (even $R(c)$-planar) for all $p \geqslant 2$. (We do not give a precise definition of "quasi-planar" in this paper (for this look at *op cit.*), but the intuitive meaning of "L is quasi-$R(c)$-planar" is clear from the following: If one looks at L, L "seems" to be $R(c)$-planar, e.g., there are no crossings-over of edges and all faces define convex sets, but some faces may be collapsed to straight line segments or even to points. For any $\varepsilon > 0$ we then have a planar layout L' of G (approximating L', such that the distance between corresponding nodes in L' and L is at most ε.)

We cannot hope to prove, that $L_{\text{bal}}^{\text{opt}}$ is $R(c)$-planar in the general case. (See the example of Fig. 5: in the optimal balanced layout two faces are collapsed to straight line segments, one edge is degenerated to a point and two pairs of edges are collapsed on each other.) But since the limit of quasi-$R(c)$-planar layouts is quasi-$R(c)$-planar, we get the following interesting result with the help of Theorem 1.

THEOREM 2. *The optimally balanced layout* $L_{\text{bal}}^{\text{opt}}$ *of a* $R(c)$*-planar graph is quasi-*$R(c)$*-planar.*

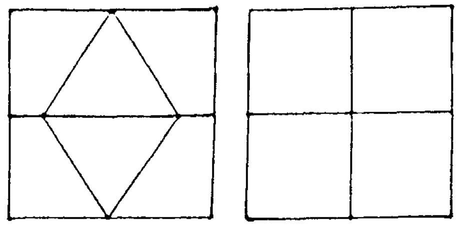

FIG. 5. In the left-hand figure L_p is $R(c)$-planar $\forall p \geq 2$; $L_{\mathrm{bal}}^{\mathrm{opt}}$ is not $R(c)$-planar in the right-hand figure.

In the introduction we gave a method to approximate the optimally balanced layout (with help of Theorem 1 and the approximation algorithms for L_p). Another possible method is to use a "divide and conquer" strategy based on the skeletons of the graph. Skeletons have the following nice property: The "optimal balanced layout" of skeleton i is independent of the existence or layout of skeletons j for $j > i$ (see Fig. 3).

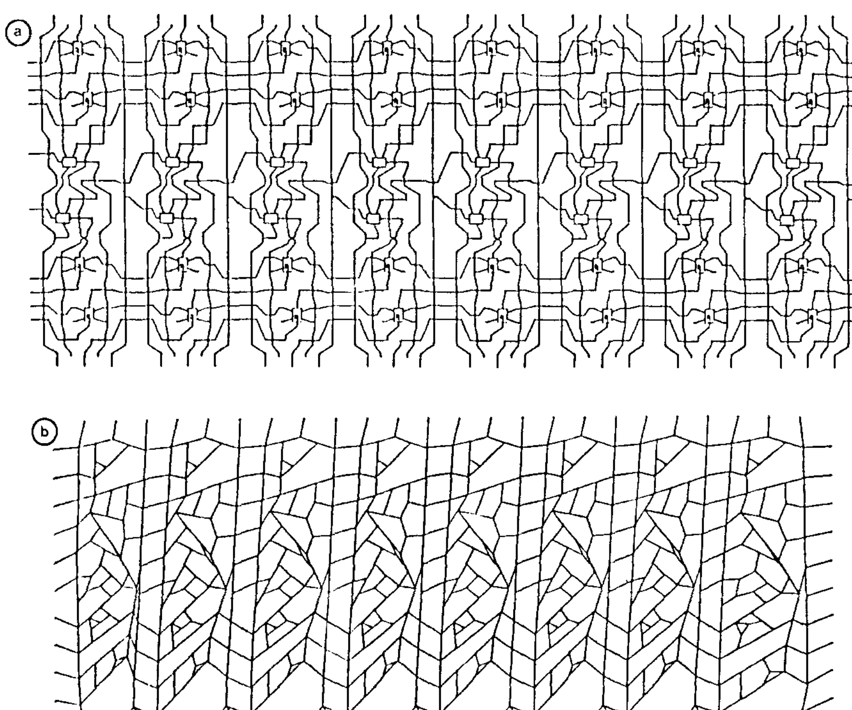

FIG. 6. (a). Graph with fixed boundary for the fast multiplier. (b). Optimal balanced layout for the multiplier.

More details and how to compute the skeletons will be given in a forthcoming paper by the second author.

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